

Quantum wave equations in curved space-time from wave mechanics

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Abstract

Alternative versions of the Klein-Gordon and Dirac equations in a curved spacetime are got by applying directly the classical-quantum correspondence.

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The usual way to write the wave equations of relativistic quantum mechanics in a curved spacetime is by *covariantization*: the searched equation in curved spacetime should coincide with the flat-spacetime version in coordinates where the connection cancels at the event X considered. This is connected with the *equivalence principle*. For the Dirac equation with standard (spinor) transformation, this procedure leads to the Dirac-Fock-Weyl (DFW) equation, which does *not* obey the equivalence principle. Alternatively, in this work we want to *apply directly the classical-quantum correspondence*.

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The latter results [1] from two mathematical facts [2]. **i)** There is a one-to-one correspondence between a (2nd-order, say) linear differential operator:

$$P\psi \equiv a_0(X) + a_1^\mu(X)\partial_\mu\psi + a_2^{\mu\nu}(X)\partial_\mu\partial_\nu\psi, \quad (1)$$

and its *dispersion equation*, a polynomial equation for covector \mathbf{K} :

$$\Pi_X(\mathbf{K}) \equiv a_0(X) + i a_1^\mu(X)K_\mu + i^2 a_2^{\mu\nu}(X)K_\mu K_\nu = 0, \quad (2)$$

the latter arising when one looks for “locally plane-wave” solutions [1]: $\psi(X) = A \exp[i\theta(X)]$, with $\partial_\nu K_\mu(X_0) = 0$, where $K_\mu \equiv \partial_\mu\theta$. The correspondence from (2) to (1) is $K_\mu \rightarrow \partial_\mu/i$. **ii)** The propagation of the spatial wave covector $\mathbf{k} \equiv (K_j)$ ($j = 1, 2, 3$) obeys a *Hamiltonian system*:

$$\frac{dK_j}{dt} = -\frac{\partial W}{\partial x^j}, \quad \frac{dx^j}{dt} = \frac{\partial W}{\partial K_j} \quad (j = 1, 2, 3), \quad (3)$$

where $W(\mathbf{k}; X)$ is the *dispersion relation*, got by solving $\Pi_X(\mathbf{K}) = 0$ for the frequency $\omega \equiv -K_0$. Wave mechanics (classical trajectories=skeleton of a wave pattern) means that the classical Hamiltonian is $H = \hbar W$. The classical-quantum correspondence follows [1] by substituting $K_\mu \rightarrow \partial_\mu/i$.

This analysis shows [3] that *the classical-quantum correspondence needs using preferred classes of coordinate systems*: the dispersion polynomial $\Pi_X(\mathbf{K})$ and the condition $\partial_\nu K_\mu(X) = 0$ stay invariant only inside any class of “infinitesimally-linear” coordinate systems, connected by changes satisfying, at point X considered, $\frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\nu} = 0$, $\mu, \nu, \rho \in \{0, \dots, 3\}$. One such class is that of locally-geodesic coordinate systems at X for metric \mathbf{g} : $g_{\mu\nu, \rho}(X) = 0$, $\mu, \nu, \rho \in \{0, \dots, 3\}$. Another class occurs if there is a (physically) preferred reference frame: that made of changes which are internal to this frame. Assuming one class or the other gives distinct wave equations.

We may now apply this correspondence to a relativistic particle, *also in a curved space-time* [3]. In each coordinate system, the energy component p_0 of the 4-momentum defines a classical Hamiltonian $H \equiv -p_0$ satisfying

$$g^{\mu\nu} p_\mu p_\nu - m^2 = 0 \quad (c = 1). \quad (4)$$

The dispersion equation associated with this by wave mechanics is

$$g^{\mu\nu} K_\mu K_\nu - m^2 = 0 \quad (\hbar = c = 1). \quad (5)$$

Applying directly the correspondence $K_\mu \rightarrow \partial_\mu/i$ to it, leads to the Klein-Gordon equation. Instead, one may try a *factorization*:

$$\Pi_X(\mathbf{K}) \equiv (g^{\mu\nu}(X)K_\mu K_\nu - m^2)\mathbf{1} = [\alpha(X) + i\gamma^\mu(X)K_\mu][\beta(X) + i\zeta^\nu(X)K_\nu]. \quad (6)$$

Identifying coefficients in (6) (with noncommutative algebra), and then substituting $K_\mu \rightarrow \partial_\mu/i$, leads to the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \quad \text{with } \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1}. \quad (7)$$

Assume the first class (locally-geodesic systems). Then Eq. (7), derived in any system of that class, rewrites in a *general* coordinate system as:

$$(i\gamma^\nu D_\nu - m)\psi = 0, \quad (D_\nu \psi)^\mu \equiv \psi^\mu_{;\nu} \equiv \partial_\nu \psi^\mu + \Gamma^\mu_{\sigma\nu} \psi^\sigma. \quad (8)$$

(The $\Gamma^\mu_{\sigma\nu}$'s are the Christoffel symbols of \mathbf{g} .) With the second class (preferred-frame systems), a different (preferred-frame) equation is got. These two equations are also distinct from the standard, DFW equation.

References

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